# Appendix week 7

**1**) We have shown in the notes that, for all $n \geq 0$ and $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}. \tag{1}$$

Note that this means that

$$\begin{aligned}\binom{n}{n-r} &= \frac{n!}{(n-r)!\,(n-(n-r))!} \quad \text{by (1)},\\ &= \frac{n!}{(n-r)!r!} = \binom{n}{r}.\end{aligned}$$

i.e.

$$\binom{n}{n-r} = \binom{n}{r}. \tag{2}$$

**Question** Can you prove this result by finding a bijection from $\mathcal{P}_r(A)$ to $\mathcal{P}_{n-r}(A)$ when $|A| = n$? (Left to student on problem sheet.)

The result (2) explains the symmetry seen around the centre vertical line in Pascal's triangle.

**2**) The symmetric form of the Binomial Theorem is

$$(a+b)^n = \sum_{s=0}^{n} \sum_{\substack{t=0 \\ s+t=n}}^{n} \frac{n!}{s!t!} a^s b^t \tag{3}$$

**Question** What does this double sum mean? In fact, because of the condition $s+t=n$, (3) is **not** the double sum it appears to be. To see this look at:

$$\begin{aligned}
\sum_{s=0}^{n} \sum_{\substack{t=0 \\ s+t=n}}^{n} \frac{n!}{s!t!} a^s b^t &= \sum_{s=0}^{n} \frac{n!}{s!} a^s \left( \sum_{\substack{t=0 \\ s+t=n}}^{n} \frac{1}{t!} b^t \right) \\
&= \underbrace{\frac{n!}{0!} a^0 \left( \sum_{\substack{t=0 \\ 0+t=n}}^{n} \frac{1}{t!} b^t \right)}_{s=0 \text{ term}} + \underbrace{\frac{n!}{1!} a^1 \left( \sum_{\substack{t=0 \\ 1+t=n}}^{n} \frac{1}{t!} b^t \right)}_{s=1 \text{ term}} + \underbrace{\frac{n!}{2!} a^2 \left( \sum_{\substack{t=0 \\ 2+t=n}}^{n} \frac{1}{t!} b^t \right)}_{s=2 \text{ term}} + \ ... \\
&\quad ... \ + \underbrace{\frac{n!}{(n-1)!} a^{n-1} \left( \sum_{\substack{t=0 \\ (n-1)+t=n}}^{n} \frac{1}{t!} b^t \right)}_{s=n-1 \text{ term}} + \underbrace{\frac{n!}{n!} a^n \left( \sum_{\substack{t=0 \\ n+t=n}}^{n} \frac{1}{t!} b^t \right)}_{s=n \text{ term}} \qquad (4)
\end{aligned}$$

But because of the $s+t=n$ condition we are only picking out *one* term from each sum over $t$ . Thus (4) equals

$$\begin{aligned}
&= \frac{n!}{0!} a^0 \times \underbrace{\frac{1}{n!} b^n}_{t=n \text{ term}} + \frac{n!}{1!} a^1 \times \underbrace{\frac{1}{(n-1)!} b^{n-1}}_{t=n-1 \text{ term}} + \frac{n!}{2!} a^2 \times \underbrace{\frac{1}{(n-2)!} b^{n-2}}_{t=n-2 \text{ term}} + \ ... \\
&\quad ... \ + \frac{n!}{(n-1)!} a^{n-1} \times \underbrace{\frac{1}{1!} b^1}_{t=1 \text{ term}} + \frac{n!}{n!} a^n \times \underbrace{\frac{1}{0!} b^0}_{t=0 \text{ term}} \\
&= b^n + nab^{n-1} + \frac{n!}{2!\,(n-2)!} a^2 b^{n-2} + \ ... \ na^{n-1} b + a^n \\
&= \sum_{r=0}^{n} \binom{n}{r} b^{n-r} a^r .
\end{aligned}$$

So the "double" sum equals the single sum!

**3**) In the notes we defined functions between $\mathcal{P}(A)$ and $Fun(A, \{0,1\})$ by

$$\begin{aligned} \alpha : \mathcal{P}(A) &\rightarrow Fun(A, \{0,1\}), \\ C &\mapsto \chi_C \end{aligned}$$

for all $C \subseteq A$, and

$$\begin{aligned} \beta : Fun(A, \{0,1\}) &\rightarrow P(A), \\ f &\mapsto C_f \end{aligned}$$

for all $f : A \rightarrow \{0,1\}$. Here

$$\chi_C(a) = \begin{cases} 1 & \text{if } a \in C \\ 0 & \text{if } a \notin C, \end{cases} \tag{5}$$

and

$$C_f = \{a \in A : f(a) = 1\}. \tag{6}$$

A result unproved in the notes was:

**Theorem 1** *The functions $\alpha$ and $\beta$ defined above are inverses of each other.*

**Proof** We have to show that $\beta \circ \alpha$ is the identity map on $\mathcal{P}(A)$ and that $\alpha \circ \beta$ is the identity map on $Fun(A, \{0,1\})$.

In other words

$$\beta \circ \alpha(C) = C \quad \text{for all } C \in \mathcal{P}(A), \tag{7}$$

and

$$\alpha \circ \beta(f) = f \quad \text{for all } f \in Fun(A, \{0,1\}) \tag{8}$$

**To prove** (7) let $C \in \mathcal{P}(A)$, i.e. $C \subseteq A$. Then

$$\beta \circ \alpha(C) = \beta(\chi_C) = C_{\chi_C}.$$

Is $C_{\chi_C} = C$? By (6) we have

$$\begin{aligned} C_{\chi_C} &= \{a \in A : \chi_C(a) = 1\} \\ &= \{a \in A : a \in C\} \qquad \text{by (5)} \\ &= C. \end{aligned}$$

Hence $\beta \circ \alpha$ is the identity on $\mathcal{P}(A)$.

**To prove** (8) let $f \in Fun\,(A, \{0,1\})$. Then

$$\alpha \circ \beta\,(f) = \alpha\,(C_f) = \chi_{C_f}.$$

Is $\chi_{C_f} = f$? From (5) we have

$$\begin{aligned}
\chi_{C_f}\,(a) &= \begin{cases} 1 & \text{if } a \in C_f \\ 0 & \text{if } a \notin C_f, \end{cases} \\
&= \begin{cases} 1 & \text{if } f\,(a) = 1 \\ 0 & \text{if } f\,(a) = 0 \end{cases} \qquad \text{by definition of } C_f \\
&= f\,(a)
\end{aligned}$$

Hence $\alpha \circ \beta$ is the identity function on $Fun\,(A, \{0,1\})$.

Therefore $\alpha$ and $\beta$ are inverses. ■

**4**) Assume there exists a bijection $f : A \to B$. Extend this to a function on $\mathcal{P}\,(A)$ by the definition

$$\begin{aligned}
\overrightarrow{f} : \mathcal{P}\,(A) &\to \mathcal{P}\,(B) \\
C &\mapsto \overrightarrow{f}\,(C) = \{f\,(c) : c \in C\}.
\end{aligned}$$

So, for all $C \in \mathcal{P}\,(A)$, (i.e. $C \subseteq A$), $\overrightarrow{f}\,(C)$ is the set of all images of elements in $C$. These images lie in $B$ (since $f : A \to B$) and so $\overrightarrow{f}\,(C) \subseteq B$ and thus $\overrightarrow{f}\,(C) \in \mathcal{P}\,(B)$. Therefore $\overrightarrow{f} : \mathcal{P}\,(A) \to \mathcal{P}\,(B)$ as stated.

Since $f$ is a bijection it has an inverse $f^{-1}$. Thus we can define a function on $\mathcal{P}\,(B)$ by

$$\begin{aligned}
\overleftarrow{f} : \mathcal{P}\,(B) &\to \mathcal{P}\,(A) \\
D &\mapsto \overleftarrow{f}\,(D) = \left\{f^{-1}\,(d) : d \in D\right\},
\end{aligned}$$

So, for all $D \in \mathcal{P}\,(B)$, $\overleftarrow{f}\,(D)$ is the set of all pre-images of elements in $D$. These images lie in $A$ (since $f^{-1} : B \to A$) and so $\overleftarrow{f}\,(D) \subseteq A$ and thus $\overleftarrow{f}\,(D) \in \mathcal{P}\,(A)$. Therefore $\overleftarrow{f} : \mathcal{P}\,(B) \to \mathcal{P}\,(A)$ as stated.

**Theorem 2** *Given a bijection* $f : A \to B$ *the extensions* $\overrightarrow{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ *and* $\overleftarrow{f} : \mathcal{P}(B) \to \mathcal{P}(A)$ *as defined above are inverses of each other.*

**Proof** We have to check that $\overleftarrow{f} \circ \overrightarrow{f}$ and $\overrightarrow{f} \circ \overleftarrow{f}$ are identities on $\mathcal{P}(A)$ and $\mathcal{P}(B)$ respectively.

**To prove that** $\overleftarrow{f} \circ \overrightarrow{f}$ **is an identity on** $\mathcal{P}(A)$

Let $C \in \mathcal{P}(A)$ be given. Then

$$\begin{aligned} \overleftarrow{f} \circ \overrightarrow{f}(C) &= \overleftarrow{f}\left(\overrightarrow{f}(C)\right) \\ &= \overleftarrow{f}(\{f(c) : c \in C\}) \\ &= \left\{f^{-1}(f(c)) : c \in C\right\} \\ &= \{c : c \in C\} \\ &= C. \end{aligned}$$

Hence $\overleftarrow{f} \circ \overrightarrow{f}$ is the identity on $\mathcal{P}(A)$.

**To prove that** $\overrightarrow{f} \circ \overleftarrow{f}$ **is an identity on** $\mathcal{P}(B)$

Let $D \in \mathcal{P}(B)$ be given. Then

$$\begin{aligned} \overrightarrow{f} \circ \overleftarrow{f}(D) &= \overrightarrow{f}\left(\overleftarrow{f}(D)\right) \\ &= \overrightarrow{f}\left(\left\{f^{-1}(d) : d \in D\right\}\right) \\ &= \left\{f\left(f^{-1}(d)\right) : d \in D\right\} \\ &= \{d : d \in D\} \\ &= D. \end{aligned}$$

Hence $\overrightarrow{f} \circ \overleftarrow{f}$ is the identity on $\mathcal{P}(B)$. ■

From this result we immediately deduce that if $|A| = |B|$ (which implies the existence of a bijection between $A$ and $B$) then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. With an extra observation we showed in the notes that $|\mathcal{P}_r(A)| = |\mathcal{P}_r(B)|$ for all $0 \le r \le |A|$. This means that the definition of the Binomial number is independent of the set $A$ chosen in the definition.

**5**) Within the proof of

$$|\mathcal{P}_r(A)| = |\mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})|$$

we define maps

$$\begin{aligned} \alpha &: \mathcal{P}_r(A) \to \mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\}) \\ C &\mapsto \begin{cases} C \setminus \{a\} & \text{if } a \in C \\ C & \text{if } a \notin C, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \beta &: \mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\}) \to \mathcal{P}_r(A) \\ D &\mapsto \begin{cases} D & \text{if } |D| = r \\ D \cup \{a\} & \text{if } |D| = r-1. \end{cases} \end{aligned}$$

The first thing done in the proof was to show that

$$\operatorname{Im} \alpha \subseteq \mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\}) \quad \text{and} \quad \operatorname{Im} \beta \subseteq \mathcal{P}_r(A).$$

But what wasn't shown in the lectures was that $\alpha$ and $\beta$ are inverses.

**Theorem 3** *The functions $\alpha$ and $\beta$ defined above are inverses of each other.*

**Proof** This requires showing that $\beta \circ \alpha$ is the identity on $\mathcal{P}_r(A)$ and $\alpha \circ \beta$ is the identity on $\mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})$.

**Looking first at** $\beta \circ \alpha$ let $C \in \mathcal{P}_r(A)$ be given. There are two cases.

i . If $a \in C$ then $\alpha(C) = C \setminus \{a\}$. But then

$$\begin{aligned} \beta(\alpha(C)) &= \beta(C \setminus \{a\}) \\ &= (C \setminus \{a\}) \cup \{a\} \quad \text{since } |C \setminus \{a\}| = r-1 \\ &= C. \end{aligned}$$

ii . If $a \notin C$ then $\alpha(C) = C$ and

$$\beta(\alpha(C)) = \beta(C) = C,$$

since $|C| = r$. So in both cases $\beta(\alpha(C)) = C$, i.e. $\beta \circ \alpha(C) = C$. True for all $C \in \mathcal{P}_r(A)$ means that $\beta \circ \alpha$ is the identity on $\mathcal{P}_r(A)$.

**Next looking at** $\alpha \circ \beta$ let $D \in \mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})$ be given. Again we have two cases (and since the union is disjoint there is no overlap in the cases).

i . If $|D| = r - 1$ then $\beta(D) = D \cup \{a\}$. But then

$$\begin{aligned} \alpha(\beta(D)) &= \alpha(D \cup \{a\}) \\ &= (D \cup \{a\}) \setminus \{a\} \quad \text{since } a \in D \cup \{a\} \\ &= D. \end{aligned}$$

ii . If $|D| = r$ then $\beta(D) = D$ and

$$\begin{aligned} \alpha(\beta(D)) &= \alpha(D) \\ &= D \quad \text{since } a \notin D \text{ (recall that } D \subseteq A \setminus \{a\}\text{)}. \end{aligned}$$

So in both cases $\alpha(\beta(D)) = D$, i.e. $\alpha \circ \beta(D) = D$. True for all $D \in \mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})$ means that $\alpha \circ \beta$ is the identity on $\mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})$.

Thus we have a bijection $\mathcal{P}_r(A) \to \mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})$. ■